The universal closed state space of an open TFT

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August 31, 2011

In this short note we explain, using a toy model, some of the ideas behind the following theorem of Costello:

Theorem 0.1. *[1, Theorem A] Let Φ be an open TCFT, and let $C_H(\Phi)$ be its Hochschild chain-complex. If the natural pairing on $C_H(\Phi)$ is homologically non-degenerate then the pair $(C_H(\Phi), \Phi)$ form the homotopy-universal open-closed TCFT whose open sector is Φ.*

A TCFT (=*Topological Conformal Field Theory*) is an abstraction of a particular kind of 2-dimensional quantum field theory, it depends on the full structure of the topology of the moduli space of Riemann surfaces. We will describe a much simpler object called a strictly topological 2d TFT, which uses only topological surfaces (i.e. π_0 of the moduli space). The extra ideas and technicalities necessary for the full TCFT case as they can be found in the above paper.

Fix a set Λ whose elements will be called *branes*. A (strictly topological) *string world-sheet* is the following data:

- A 2d oriented manifold with boundary Σ.

- An ordered set of oriented components of the boundary $\partial\Sigma$. These are the *closed strings*.

- An ordered set of disjoint oriented intervals embedded in the remaining part of $\partial\Sigma$. These are the *open strings*.

- The pieces of $\partial\Sigma$ that are neither open nor closed strings are called *free boundaries*, they can be either intervals or circles. Each free boundary is labelled with a brane.

The closed and open strings may be oriented in a way that is either compatible with or opposite to the orientation on Σ. The former are called *incoming* and
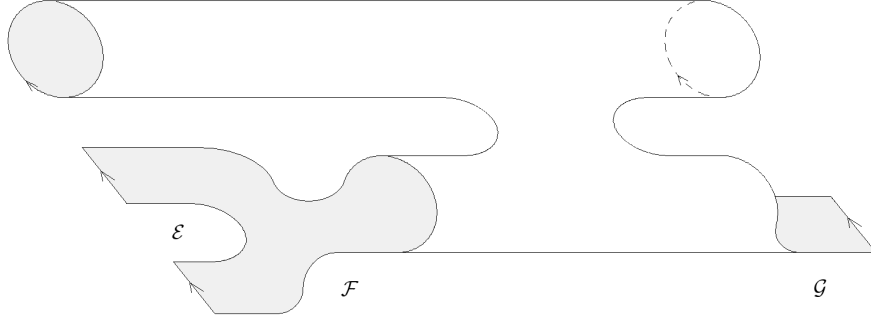
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Figure 1: A string world-sheet

the later *outgoing*. Notice that every open string starts at one brane and ends at another (or possibly the same) brane.

Figure 1 shows an example world-sheet. It has three incoming strings, of which one is closed and two are open (shown on the left), and two outgoing strings, one closed and one open (shown on the right). There are three free boundaries labelled by branes $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$.

String world-sheets form the morphisms of a category **Strng**. An object of **Strng** is a pair consisting of an ordered set of circles and an ordered set of oriented intervals, each of which is labelled at both ends by a brane. A world-sheet is a morphism between the object formed by its incoming strings and the object formed by its outgoing strings. Composition of morphisms in **Strng** is given by gluing the outgoing strings of one world-sheet to the incoming strings of another.

Taking the disjoint union of world-sheets gives **Strng** a monidal product. There is an object $\{\circ\}$ in **Strng** consisting of a single circle, and there is a family of objects $\{\mathcal{E} \rightarrow \mathcal{F}\}$ indexed by $\Lambda \times \Lambda$ which consist of a single interval going between two branes. Obviously these objects together freely generate the objects of **Strng** under the monoidal product.

We can also write down a set of world-sheets that generate **Strng** as a monoidal category. Firstly note that any world-sheet Σ has a dual $\Sigma^{\dagger}$ where we flip the orientations on all the strings. It is clear (e.g. by Morse theory) that the following set of world-sheets, together with their duals, generate **Strng**:

- P_c, the genus-zero surface with two incoming and one outgoing closed strings (the 'pair-of-pants').

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- D_c, the disc whose boundary is an incoming closed string.

- I_c, the cylinder with one incoming and one outgoing closed string.

- P_o, the disc with two incoming and one outgoing open strings embedded in its boundary.

- I_o, the disc with one incoming and one outgoing open sting in its boundary.

- T, the cylinder with one end an outgoing closed string and one incoming open string in the other end (the 'penny-whistle').[1]

The world-sheets in this list that have free boundaries come in families indexed by products of Λ. We also have to add 're-ordering' morphisms which consist of products of I_c and I_o but with different orders on the sets of strings at each end. Notice that the disc D_o with one incoming string is not in this list, since it can be obtained by gluing T to D_c.

As an example, the world-sheet in Fig. 1 has source $\{\mathcal{F} \to_{\mathcal{E}},\ \mathcal{E} \to_{\mathcal{F}}, \circ\}$, target $\{\mathcal{G} \to_{\mathcal{G}}, \circ\}$, and can be written as the composition

$$\Sigma = (I_c \sqcup T^\dagger P_c) \circ (P_c^\dagger \sqcup T P_o)$$

A 2d (strictly) *Topological Field Theory* (TFT) is a monoidal functor

$$\Psi : \mathbf{Strng} \to \mathbf{Vect}$$

Since Ψ is monoidal, it is determined on objects by its value on the single closed string

$$V := \Psi(\circ)$$

which is called the *closed state space*, and its value on each labelled open string

$$A(\mathcal{E}, \mathcal{F}) := \Psi(_{\mathcal{E}} \to_{\mathcal{F}})$$

which is the *open state space* between $\mathcal{E}$ and $\mathcal{F}$. Applying Ψ to our generating set of morphisms gives us various linear maps between these vector spaces, for example we have a bilinear product

$$\Psi(P_c) : V^{\otimes 2} \to V$$

These linear maps are subject to various relations coming from relations amongst world-sheets. For example, it is clear that

$$P_c \circ (P_c \sqcup I_c) \cong P_c \circ (I_c \sqcup P_c)$$

which implies that the product $\Psi(P_c)$ is associative. The totality of these relations mean that a TFT is equivalent to the following data:

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[1] In Dutch: 'Fluitje van een cent', which means something easy or having no content. This worldsheet will used later as the basis of a formal adjoint construction.

- A commutative Frobenius algebra V with product $\Psi(P_c)$, unit $\Psi(I_c)$, and trace map $\Psi(D_c)$.

- A linear category A, whose set of objects is Λ, with composition

$$\Psi(P_o) : A(\mathcal{E}, \mathcal{F}) \otimes A(\mathcal{F}, \mathcal{G}) \to A(\mathcal{E}, \mathcal{G})$$

  and identity morphisms $\Psi(I_o) \in A(\mathcal{E}, \mathcal{E})$.

- A map (the 'boundary-bulk map')

$$\Psi(T) : A(\mathcal{E}, \mathcal{E}) \to V$$

  for each brane $\mathcal{E}$, such that

  1. The bilinear pairing $\Psi(D_c) \circ \Psi(T) \circ \Psi(P_o)$ on A is non-degenerate and cyclically-invariant.
  2. The adjoint $\Psi(T)^\dagger$ (with respect to the pairings on A and V) is a map of algebras
  3. For any two branes $\mathcal{E}$ and $\mathcal{F}$

  $$\Psi(P_c) \circ (\Psi(T) \otimes \Psi(T)) = \sum_i \Psi(T) \circ \Psi(P_o) \circ (\Psi(P_o)(-, e_i) \otimes \Psi(P_o)(-, \epsilon_i))$$

  where $\{e_i\}$ and $\{\epsilon_i\}$ are dual bases of $A(\mathcal{E}, \mathcal{F})$ and $A(\mathcal{F}, \mathcal{E})$.

Condition (1) says that A is a *Calabi-Yau* category (it is the same as the structure that makes V a Frobenius algebra, unfortunately 'Frobenius category' means something else). Condition (3) is called the *Cardy condition* and maybe needs a little explanation. It comes from the topological equivalence of the two world-sheets in Fig. 2 plus the fact that the copairing on A (the adjoint to the pairing) is given by $\sum_i e_i \otimes \epsilon_i$.

A little more physics terminology: V and A are called the *closed* and *open* *sectors* of the theory, and if Σ is a world-sheet with no outgoing strings then $\Psi(\Sigma)$ is called a *correlator*. For example $\Psi(D_c)$ is the *1-point* closed correlator on a disc. Notice that the 1-point open correlator on a disc $\Psi(D_o)$ factors through the boundary-bulk map.

Now we come to the key construction from the point of view of this paper. There is a subcategory $\mathbf{Strng}_o \subset \mathbf{Strng}$ where we don't allow any closed strings, a functor from this subcategory to $\mathbf{Vect}$ is called an *open* TFT, by the above this is the same thing as a Calabi-Yau category A. We will show how, given one extra hypothesis, we can canonically extend any open TFT to a full TFT.

Let A be a TFT, and consider the vector space
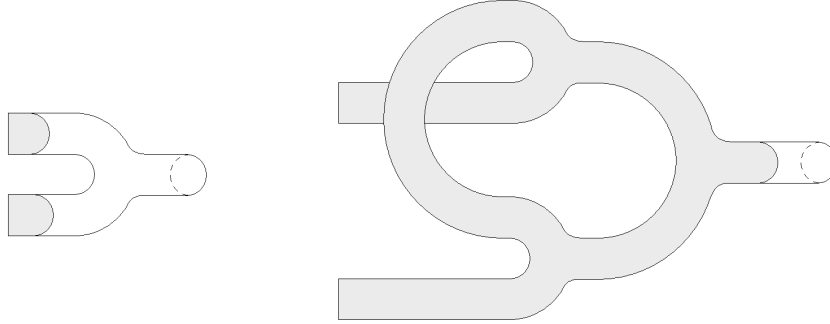
$$V = A \,/\, [A, A]$$

Figure 2: The Cardy condition

We define a pairing on V by

$$\langle [\alpha], [\beta] \rangle := \mathrm{Tr}(x \mapsto \alpha x \beta)$$

where $\alpha \in A(\mathcal{E}, \mathcal{E})$, $\beta \in A(\mathcal{F}, \mathcal{F})$, and the trace is of the given endomorphism of $A(\mathcal{E}, \mathcal{F})$. It is easy to check that this is well-defined.

Lemma 0.2. *Let A be an open TFT, and let V be the above vector space. Assume that the pairing $\langle , \rangle$ is non-degenerate. Then V is the canonical closed state space for A.*

By 'canonical closed state space', we mean that (V, A) form a TFT, and that taking A to (V, A) gives an adjoint to the forgetful functor that sends a TFT to its open sector. As we explained above this is just a baby version of Costello's theorem (Thm. 0.1).

Proof. Let $\mathcal{V}$ be the set of world-sheets whose incoming strings are all open, and that have exactly one outgoing closed string and no outgoing open strings. This set carries a left action of the category $\mathbf{Strng}_o$ by gluing on purely open worldsheets. Note that by definition A carries a right action of $\mathbf{Strng}_o$. We define

$$V = A \otimes_{\mathbf{Strng}_o} \mathcal{V}$$

To get V we take the free vector space spanned by world-sheets in $\mathcal{V}$ with their open strings labelled by elements of A, then quotient by relations coming from the action of $\mathbf{Strng}_o$ on either side. An example relation is shown in Fig. 3(a). It is clear that by applying successive relations we can reduce everything to the form $\alpha \otimes T$ for some $\alpha \in A$ (as in the RHS of Fig. 3(a)). However there is still the residual relation shown in Fig. 3(b), so in fact $V = A \,/\, [A, A]$.
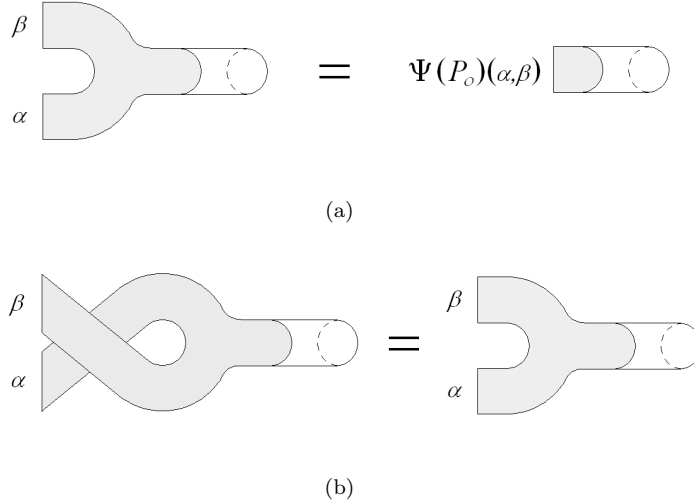
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(a)

(b)

Figure 3: Some relations in V

Now consider the cylinder with two incoming closed strings $(P_c \circ D_c)$. This induces a pairing on V, since given two elements of V we can glue them into the ends of the cylinder and get a correlator which we can evaluate in the open sector. This is the pairing $\langle, \rangle$, it is well-defined by the definition of V. By the same reasoning, we can take any correlator, label it with elements of A and V and then evaluate it using just the structure of A. Since by assumption the pairing on V is non-degenerate, this in fact defines a linear map for all world-sheets, because we can identify V and $V^\vee$. Thus (V, A) forms a TFT. $\square$

Finally we will say a few words about the more sophisticated case of TCFTs dealt with in [1]. Firstly, we let our world-sheets be Riemann surfaces (with boundary) instead of topological surfaces, and since these come in moduli spaces this produces a topological category. Applying a suitable chain functor (one that computes topological homology and is compatible with the gluing structure) we get a dg-category. A TCFT is a functor from this dg-category to **dgVect**.

The open sector A of a TCFT is in general a Calabi-Yau A_∞-category, but up to homotopy we can represent it as a Calabi-Yau dg-category. Since this is just a 'derived' analogue of a CY linear category it is then plausible that the canonical closed sector should be a derived version of $A \,/\, [A, A]$. This is precisely the Hochschild chain complex.

References

[1] Kevin Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214.